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ON THE RECIPROCAL QUARTIC EQUATION.

By DR. R. L. BERGER.

In this paper it is proposed to determine the Galois group of the reciprocal quartic equation,

$$x^4-ax^3+bx^2-ax+1=0\dots(1),$$

for the domain of rationality $R(1)$. We shall establish the conditions for which the group of the equation is transitive, and hence the condition that the equation is irreducible; and consider the possible intransitive groups when these conditions are not fulfilled.

Calling the roots of (1) α_0, α_1, β_0, β_1; α_0 the reciprocal of β_0, and α_1 the reciprocal of β_1,

$$\alpha_0\beta_0=\alpha_1\beta_1=1\dots(2).$$

If we assume that the roots of (1) are distinct, the Galois group of (1) is a subgroup of the group,

$$\begin{aligned}G_8\equiv[1;\ &(\alpha_0\beta_0);\ (\alpha_1\beta_1);\ (\alpha_0\beta_0)(\alpha_1\beta_1);\ (\alpha_0\alpha_1)(\beta_0\beta_1);\\ &(\alpha_0\beta_1)(\alpha_1\beta_0);\ (\alpha_0\alpha_1\beta_0\beta_1);\ (\alpha_1\alpha_0\beta_1\beta_0)].\end{aligned}$$

For, any substitution not in G_8 leaves the relation (2) unaltered only if the equation (1) has a pair of equal roots. The group G_8 has only two transitive subgroups, viz:

$$C_4\equiv[1;\ (\alpha_0\alpha_1\beta_0\beta_1);\ (\alpha_1\alpha_0\beta_1\beta_0);\ (\alpha_0\beta_0)(\alpha_1\beta_1)],$$
$$\text{and } G_4\equiv[1;\ (\alpha_0\alpha_1)(\beta_0\beta_1);\ (\alpha_0\beta_1)(\alpha_1\beta_0);\ (\alpha_0\beta_0)(\alpha_1\beta_1)].$$

Hence, if we impose such conditions upon the coefficients of (1) that its group is either G_8, G_4, or C_4 we have the necessary and sufficient conditions that the equation be irreducible. For this purpose it is necessary to compute the values in terms of the coefficients of functions belonging to each of the following subgroups of G_8:

$$G_4;\ C_4;\ H_4\equiv[1;\ (\alpha_0\beta_0);\ (\alpha_1\beta_1);\ (\alpha_0\beta_0)(\alpha_1\beta_1)].$$
$$G'_2\equiv[1;\ (\alpha_0\alpha_1)(\beta_0\beta_1)];\ G''_2\equiv[1;\ (\alpha_0\beta_1)(\alpha_1\beta_0)].$$

From the equation (1); $\alpha_0+\beta_0+\alpha_1+\beta_1=a\dots(3)$,

$$(\alpha_0+\beta_0)(\alpha_1+\beta_1)=b-2\dots(4).$$

By means of (3) and (4) we easily find;

$\phi \equiv (\alpha_0+\beta_0) - (\alpha_1+\beta_1) = \sqrt{[2-b+\frac{1}{4}a^2]}$, belonging to H_4... (5),
$\psi \equiv (\alpha_0-\beta_0)(\alpha_1-\beta_1) = \sqrt{[(1+\frac{1}{2}b)^2-a^2]}$, belonging to G_4... (6),
$\phi.\psi$ belonging to C_4... (7).

With the aid of (5) and (6);

$\phi' \equiv \alpha_0+\alpha_1 = \sqrt{\{\frac{1}{4}a^2-(1+\frac{1}{2}b)-\sqrt{[(1+\frac{1}{2}b)^2-a^2]}\}}$, belonging to G'_2... (8),
$\phi'' \equiv \alpha_0+\beta_1 = \sqrt{\{\frac{1}{4}a^2-(1+\frac{1}{2}b)+\sqrt{[(1+\frac{1}{2}b)^2-a^2]}\}}$, belonging to G''_2... (9),
$\phi''' \equiv (\alpha_0-\beta_1)^2 = \sqrt{(b^2-4)}$ (when $a=0$), belonging to G''_2... (10).

The values for ϕ and ψ are given up to a rational factor; those for ϕ', ϕ'', ϕ''' up to a rational term.

In the study of the cases where the group is intransitive we shall make use of the two additional functions:

$x_0 \equiv \alpha_0-\beta_0 = +\sqrt{\{\frac{1}{2}a^2-(b+2)+a\sqrt{[\frac{1}{4}a^2+2-b]}\}}$,
belonging to $H_2 \equiv [1;\ (\alpha_1\beta_1)]$... (11),
$x_1 \equiv \alpha_1-\beta_1 = +\sqrt{\{\frac{1}{2}a^2-(b+2)-a\sqrt{[\frac{1}{4}a^2+2-b]}\}}$,
belonging to $H'_2 \equiv [1;\ (\alpha_0\beta_0)]$... (12).

From the definition of the Galois group of an equation for a domain of rationality R it follows that if a rational function of the roots of the equation belonging to a group H has a value not in R the group of the equation is not contained in H. Hence we need to consider subgroups of a group H only when a function belonging to H has its value in R.

The group of (1) will depend on the character of the functions (5)-(12) and we have in the following three cases the conditions for which the group G is transitive:

1) ϕ irrational, ψ irrational, and $\phi.\psi$ irrational.

2) ϕ irrational, ψ irrational, and $\psi.\phi$ rational.

$\therefore G=C_4$ or a subgroup of C_4. But its subgroups are excluded as they are subgroups of H_4 to which ϕ belongs. $\therefore G=C_4$.

3) ϕ irrational, ψ rational. Then $G=G_4$, G'_2 or G''_2.

We distinguish two cases:

I) $a\neq 0$. If ϕ' and ϕ'' are irrational, *i. e.*, if the two values $+\sqrt{\{\frac{1}{4}a^2-(1+\frac{1}{2}b)\pm\sqrt{[(1+\frac{1}{2}b)^2-a^2]}\}}$ are irrational, G'_2 and G''_2 are excluded, and $G=G_4$.

II) $a=0$. In this case $\phi''=0$ and we use $\phi'''=\sqrt{(b^2-4)}\neq 0$ to exclude G'_2. $\phi'''\neq 0$, since $b=2$ makes ϕ rational and $b=-2$ makes the equation 1) have a pair of equal roots. If then ϕ' is irrational, *i. e.*, in this case, $\sqrt{(-2-b)}$ is irrational and if $\sqrt{(b^2-4)} \equiv \phi'''$ is irrational, $G=G_4$.

These are the only cases in which the group G is transitive. Therefore we conclude that the necessary and sufficient conditions for the irreducibility of (1) are, if $a\neq 0$,

I) $\surd(2-b+\frac{1}{4}a^2)$ irrational, and $\surd[(1+\frac{1}{2}b)^2-a^2]$ irrational.

II) $\surd(2-b+\frac{1}{4}a^2)$ irrational, $\surd[(1+\frac{1}{2}b)^2-a^2$ rational, and $\surd\{\frac{1}{4}a^2-(1+\frac{1}{2}b)\pm\surd[(1+\frac{1}{2}b)^2-a^2]\}$ irrational.

III) When $a=0$, $\surd(-2-b)$, $\surd(2-b)$, $\surd(b^2-4)$ all irrational. When these conditions are not fulfilled the equation is reducible and its group may be any of the intransitive subgroups of G_8. By examining the values of the set of functions (5)-(12) for any particular case the groups G can be determined. The following cases arise:

1) ϕ rational, ψ irrational. Hence, $G=H_4$, H_2, or H'_2, dependent upon rational or irrational nature of x_0, x_1. Since ψ is irrational, x_0 and x_1 are not both rational. Therefore $G\neq G_1$.

2) ϕ rational, ψ rational. Hence, $G=G_2\equiv[1;\ (\alpha_0\beta_0)(\alpha_1\beta_1)]$, the greatest common subgroup of G_4, H_4 and C_4, or if all remaining functions are rational, then $G=G_1$.

3) ϕ irrational, ψ rational. Hence, $G=G'_2$ or G''_2, according as ϕ' or ϕ'' is rational. The case ϕ', ϕ'' both irrational has been included in the previous discussion.

DEPARTMENTS.

SOLUTIONS OF PROBLEMS.

GEOMETRY.

322. Proposed by FRANK LOXLEY GRIFFIN, S. M., Ph. D., Instructor in Mathematics, Williams College.

Find all surfaces such that the normal lengths intercepted by the three coordinate planes are in constant ratios for all points.

Solution by G. B. M. ZERR, A. M., Ph. D., 4243 Girard Avenue, Philadelphia, Pa.

$$\left.\begin{aligned} n-y+\frac{dz}{dy}(s-z)=0, \\ m-x+\frac{dz}{dx}(s-z)=0, \end{aligned}\right\}$$ is the equation to the normal, where (x, y, z) is the point on the surface.

$(x+zdz/dx,\ y+zdz/dy,\ 0)$, where the normal meets the xy plane.

$(x-ydy/dx,\ 0,\ ydy/dz+z)$, where the normal meets the xz plane.

$(0,\ y-xdx/dy,\ xdx/dz+z)$, where the normal meets the yz plane.

Then